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LETTER TO THE EDITOR

A new two-dimensional lattice model that is 'consistent around a cube'

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Abstract
For two-dimensional lattice equations, one definition of integrability is that the model can be naturally and consistently extended to three dimensions, i.e. that it is 'consistent around a cube' (CAC). As a consequence of CAC one can construct a Lax pair for the model. Recently Adler, Bobenko and Suris conducted a search based on this principle and certain additional assumptions. One of those assumptions was the 'tetrahedron property', which is satisfied by most known equations. We present here one lattice equation that satisfies the consistency condition but does not have the tetrahedron property. Its Lax pair is also presented, and some basic properties are discussed.

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1. Introduction

Within the field of integrable dynamics, there have recently been interesting developments in the study of integrable *difference* equations (an overview of the topic can be obtained from the proceedings mentioned in [1]). These include both the discussion on what is a proper definition of integrability, or whether the different suggested definitions actually agree or not. Numerous integrable (in some sense) difference equations have been proposed by discretizing known ODEs and PDEs in a way that retains some good properties. Once a definition of integrability has been proposed one can also try to search for all equations having the chosen property.

In this letter we consider integrable difference equations defined on a two-dimensional lattice. We assume that the lattice is rectangular and infinite, and concentrate on some representative square in it (see figure 1). (We normally use subscripts in square brackets to indicate shifts in the indices, but for a composite expression, say A, we sometimes also use $\tilde{A}$ for shift in the 1-direction and $\hat{A}$ for a shift in the 2-direction.) In general the map is given in

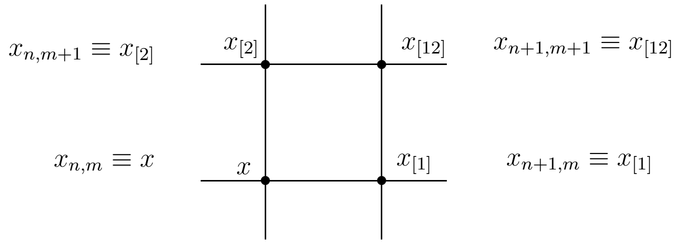

Figure 1. The lattice map is defined on an elementary square of the lattice.

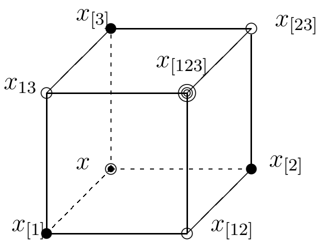

Figure 2. Given the values at the black circles, one should get a unique value for $x_{[123]}$, even though there are three possible ways to compute it.

terms of a multi-linear equation relating the four corner values in figure 1:

$$\begin{aligned} K x x_{[1]} x_{[2]} x_{[12]} &+ L_1 x x_{[1]} x_{[2]} + L_2 x x_{[1]} x_{[12]} + L_3 x x_{[2]} x_{[12]} + L_4 x_{[1]} x_{[2]} x_{[12]} + P_1 x x_{[1]} \\ &+ P_2 x_{[1]} x_{[2]} + P_3 x_{[2]} x_{[12]} + P_4 x_{[12]} x + P_5 x x_{[2]} + P_6 x_{[1]} x_{[12]} + R_1 x \\ &+ R_2 x_{[1]} + R_3 x_{[2]} + R_4 x_{[12]} + U \equiv Q_{12}(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0. \end{aligned} \tag{1}$$

Here the coefficients K, L_ν, P_ν, R_ν and U may depend on the two spectral parameters p_1, p_2. If any three of the corner values are given then the fourth one can be obtained as a rational expression of the other three. One can therefore propagate any staircase-like initial value line to cover the whole plane [2].

How should integrability be defined for such maps? In [3] the following definition of integrability ('consistency around the cube' (CAC)) was proposed (see figure 2): adjoin a third direction (therefore assuming $x = x_{n,m,k}$) and use the same map (but with different spectral parameters) also in planes corresponding to indices 1,3 and 2,3. That is, the map given in (1) contains shifts and parameters associated with directions 1,2 now the same should be done with directions 3,1 and 2,3, furthermore, on parallel-shifted planes we use identical maps. We assume that the values x, $x_{[1]}$, $x_{[2]}$, $x_{[3]}$ at the black circles in figure 2 are given, then the values at the open circles are uniquely determined using the relevant map, but the value at $x_{[123]}$ can be computed in three different ways, and they must give the same result. In other words:

$$\begin{aligned} &\text{solve } x_{[12]} \text{ from} && Q_{12}(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) = 0 \\ &\text{solve } x_{[23]} \text{ from} && Q_{23}(x, x_{[2]}, x_{[3]}, x_{[23]}; p_2, p_3) = 0 \\ &\text{solve } x_{[31]} \text{ from} && Q_{31}(x, x_{[3]}, x_{[1]}, x_{[31]}; p_3, p_1) = 0 \end{aligned}$$

then $x_{[123]}$ computed from

$$\begin{aligned} &Q_{12}(x_{[3]}, x_{[31]}, x_{[23]}, x_{[123]}; p_1, p_2) = 0 \quad \text{or} \\ &Q_{23}(x_{[1]}, x_{[12]}, x_{[31]}, x_{[123]}; p_2, p_3) = 0 \quad \text{or} \\ &Q_{31}(x_{[2]}, x_{[23]}, x_{[12]}, x_{[123]}; p_3, p_1) = 0 \end{aligned}$$

should be the same. The functions Q_{ij} could in principle be different, but are usually assumed to be identical.

The above idea (and diagram) resembles, e.g., the 3D Bianchi diagram that is obtained from consistency of Moutard transformations (see [4]), but it is used here in a different context: the diagram introduces a condition for lattice maps defined on *squares* rather than for Moutard transformations defined on *edges*. Furthermore, this CAC principle is a *constructive* definition of integrability in the sense that it leads algorithmically to a Lax pair.

The following maps are well-known examples that have the CAC property [1, 2]:

1. Lattice KdV: $(p_1 - p_2 + x_{[2]} - x_{[1]})(p_1 + p_2 + x - x_{[12]}) = p_1^2 - p_2^2$
2. Lattice MKdV: $p_1(xx_{[2]} - x_{[1]}x_{[12]}) = p_2(xx_{[1]} - x_{[2]}x_{[12]})$
3. Lattice SKdV: $(x - x_{[2]})(x_{[1]} - x_{[12]})p_2^2 = (x - x_{[1]})(x_{[2]} - x_{[12]})p_1^2$.

2. Searching for integrable lattices

Now that the definition of integrability has been given, one may ask for a listing of all integrable models. The complete classification of CAC maps is in fact a formidable open problem. The set of equations that needs to be solved can be derived from above: comparing the different forms for $x_{[123]}$ and collecting the various coefficients of x and its shifts, we get 2×375 functional equations for the coefficient functions (in practice we have often used 3×375 equations for a symmetric approach). The equations are polynomial in the coefficient functions appearing in (1), but the functions depend on different pairs of the three spectral parameters p_i, $i = 1, 2, 3$ and all three p_i appear in each equation. The equation list itself takes 35 MB to store.

In [5] the consistency equations were solved in the case where the map Q is the same on all planes, and under two additional assumptions:

- Symmetry:

$$\begin{aligned} Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p_1, p_2) &= \varepsilon Q(x, x_{[2]}, x_{[1]}, x_{[12]}; p_2, p_1) \\ &= \sigma Q(x_{[1]}, x, x_{[12]}, x_{[2]}; p_1, p_2) \qquad \varepsilon, \sigma = \pm 1. \end{aligned} \tag{2}$$

- 'Tetrahedron property': $x_{[123]}$ does not depend on x.

With these assumptions the authors were able to get a full classification resulting with nine models.

The tetrahedron assumption is indeed satisfied by most of the well-known models, but one can nevertheless ask whether it is a fundamental or essential property and whether there are any nontrivial models that do not satisfy it. It should be immediately observed that the more or less trivial models

$$x_{[1]}x_{[2]} - xx_{[12]} = 0 \qquad \text{and} \qquad y - y_{[1]} - y_{[2]} + y_{[12]} = 0 \tag{3}$$

($x = \mathrm{e}^y$) do have the CAC property but *not* the tetrahedron property, in fact one quickly finds that

$$x_{[123]} = x_{[1]}x_{[2]}x_{[3]}x^{-2} \qquad \text{and} \qquad y_{[123]} = y_{[1]} + y_{[2]} + y_{[3]} - 2y \tag{4}$$

respectively. One may now ask whether these models have nontrivial extensions.

The symmetry assumptions (2) lead to two possibilities w.r.t. σ, the one with $\sigma = -1$ is

$$\begin{aligned} Q = a_1(p_1, p_2)(xx_{[1]}x_{[2]} - xx_{[2]}x_{[12]} - xx_{[1]}x_{[12]} + x_{[1]}x_{[2]}x_{[12]}) \\ + a_2(p_1, p_2)(xx_{[12]} - x_{[1]}x_{[2]}) + a_3(p_1, p_2)(x - x_{[1]} - x_{[2]} + x_{[12]}). \end{aligned} \tag{5}$$

In [5] it was observed that in this symmetry class there are no integrable cases with the tetrahedron property, which is true. The ansatz (5) forms, fortunately, a rather simple class and a direct computation shows that it contains one new integrable map

$$xx_{[1]}x_{[2]} - xx_{[1]}x_{[12]} - xx_{[12]}x_{[2]} + x_{[1]}x_{[2]}x_{[12]} + (x_{[1]}x_{[2]} - xx_{[12]})(p_1 + p_2) \\ - p_1 p_2(x - x_{[1]} - x_{[2]} + x_{[12]}) = 0 \tag{6}$$

with two parameters. Since it can also be written as

$$\begin{matrix} x & x_{[1]} & x_{[2]} \\ -x & x_{[1]} & x_{[12]} \\ -x & x_{[2]} & x_{[12]} \\ +x_{[1]} & x_{[2]} & x_{[12]} \end{matrix} \quad \begin{matrix} \\ +(p+q) & x_{[1]} & x_{[2]} \\ -(p+q) & x & x_{[12]} \\ \end{matrix} \quad \begin{matrix} -pq & x_{[12]} \\ +pq & x_{[2]} \\ +pq & x_{[1]} \\ -pq & x \end{matrix} = 0$$

we call it 'the bow-tie model'. Model (6) is, however, still Möbius-equivalent to the first model in (3) by $x \mapsto -(p_1 x + p_2)/(x + 1)$ (although this transformation does not trivialize the maps on the other planes, since it depends explicitly on p_1, p_2). Therefore, we decided to search for possible extensions. This was done perturbatively, with completely general first- and second-order extensions to the bow-tie map (this work was done with REDUCE 3.7 [6], and 1 GB memory). In this approach it is not necessary to specify the details of the parameter dependence—indeed the result was surprising in this respect. From the outcome of this exercise we noted certain weaker symmetry properties among the coefficient functions, and when the (nonperturbative) computations were done with these properties (and translational invariance) the following solution was found:

$$\begin{aligned} Q(x, x_{[1]}, x_{[2]}, x_{[12]}; e_1, o_1; e_2, o_2) &\equiv xx_{[1]}x_{[2]}(o_1 - o_2) - xx_{[1]}x_{[12]}(e_1 - o_2) \\ &\quad - xx_{[2]}x_{[12]}(o_1 - e_2) + x_{[1]}x_{[2]}x_{[12]}(e_1 - e_2) + (x_{[1]}x_{[2]} - xx_{[12]})(e_1 o_1 - e_2 o_2) \\ &\quad + (xx_{[2]}o_1 - x_{[1]}x_{[12]}e_1)(e_2 - o_2) + (x_{[2]}x_{[12]}e_2 - xx_{[1]}o_2)(e_1 - o_1) \\ &\quad - x_{[12]}e_1 e_2(o_1 - o_2) + x_{[2]}e_2 o_1(e_1 - o_2) + x_{[1]}e_1 o_2(o_1 - e_2) - xo_1 o_2(e_1 - e_2) \\ &= 0. \end{aligned} \tag{7}$$

This new model is the main result of this letter. It is interesting to note that in (7) there are *two* parameters o_i, e_i in each direction, and that if $e_i = o_i$ then $o_1 - o_2$ factors out leaving (6). The parameters with different indices must all be different, otherwise the map factorizes. This model obeys the CAC property, in fact one finds $x_{[123]} = N_{[123]}/D_{[123]}$, where

$$\begin{aligned} N_{[123]} = &-x(x_{[1]} + o_1)(x_{[2]} + o_2)(x_{[3]} + o_3)(o_1 - o_2)(o_2 - o_3)(o_3 - o_1) \\ &+ (x_{[1]} + o_1)(x_{[2]} + o_2)(x_{[3]} + o_3)[(e_1 e_2 + e_3 o_3)o_3(o_1 - o_2) \\ &+(e_2 e_3 + e_1 o_1)o_1(o_2 - o_3) + (e_3 e_1 + e_2 o_2)o_2(o_3 - o_1)] \\ &+ (x + e_3)(x_{[1]} + o_1)(x_{[2]} + o_2)o_3(o_1 - o_2)(e_2 - o_3)(o_3 - e_1) \\ &+ (x + e_1)(x_{[2]} + o_2)(x_{[3]} + o_3)o_1(o_2 - o_3)(e_3 - o_1)(o_1 - e_2) \\ &+ (x + e_2)(x_{[3]} + o_3)(x_{[1]} + o_1)o_2(o_3 - o_1)(e_1 - o_2)(o_2 - e_3) \\ D_{[123]} = &\ (x_{[1]} + o_1)(x_{[2]} + o_2)(x_{[3]} + o_3)[(e_1 e_2 + e_3 o_3)(o_2 - o_1) \\ &+ (e_2 e_3 + e_1 o_1)(o_3 - o_2) + (e_3 e_1 + e_2 o_2)(o_1 - o_3)] \\ &+ (x + e_3)(x_{[1]} + o_1)(x_{[2]} + o_2)(o_1 - o_2)(e_1 - o_3)(e_2 - o_3) \\ &+ (x + e_1)(x_{[2]} + o_2)(x_{[3]} + o_3)(o_2 - o_3)(e_2 - o_1)(e_3 - o_1) \\ &+ (x + e_2)(x_{[3]} + o_3)(x_{[1]} + o_1)(o_3 - o_1)(e_3 - o_2)(e_1 - o_2). \end{aligned}$$

These are symmetric under permutations of 1, 2, 3, and the explicit x-dependence demonstrates violation of the tetrahedron property of [5].

3. Symmetries

The model (7) is invariant under the Möbius transformation

$$X \mapsto \frac{\alpha X + \beta}{\gamma X + \delta} \qquad \text{for} \quad X = x, x_i, x_{ij} \text{ and } x_{ijk}$$

$$P \mapsto -\frac{\alpha P - \beta}{\gamma P - \delta} \qquad \text{for} \quad P = e_i \text{ and } o_i.$$

It has the symmetries

$$Q(x, x_{[1]}, x_{[2]}, x_{[12]}; e_1, o_1, e_2, o_2) = -Q(x, x_{[2]}, x_{[1]}, x_{[12]}; e_2, o_2, e_1, o_1) \tag{8}$$

$$Q(x, x_{[1]}, x_{[2]}, x_{[12]}; e_1, o_1, e_2, o_2) = -Q(x_{[1]}, x, x_{[12]}, x_{[2]}; o_1, e_1, e_2, o_2). \tag{9}$$

These are similar to (2) with $\varepsilon = \sigma = -1$, but note that in (9) there is an exchange between the two parameters related to direction 1. Indeed, it seems that the extension of the parameter space from one to two dimensional (in each direction of the lattice) allows the introduction of a nontrivial exchange symmetry, resulting in the nontrivial model. (Recall that if $e_i = o_i$ the model simplifies to (6).)

The map (7) can also be written in the following rational form

$$\frac{x + e_2}{x + e_1}\frac{x_{[12]} + o_2}{x_{[12]} + o_1} = \frac{x_{[1]} + e_2}{x_{[1]} + o_1}\frac{x_{[2]} + o_2}{x_{[2]} + e_1}. \tag{10}$$

(The Bäcklund equation presented in equation (5.11) of [7] is similar, but the distribution of the terms is in fact different.) A form equivalent to (10) is

$$\frac{o_2 + x_{[12]}}{o_2 + x_{[2]}}\frac{e_2 + x}{e_2 + x_{[1]}} = \frac{o_1 + x_{[12]}}{o_1 + x_{[1]}}\frac{e_1 + x}{e_1 + x_{[2]}}$$

and comparing these one observes duality under

$$o_2 \leftrightarrow x \qquad e_2 \leftrightarrow x_{[12]} \qquad o_1 \leftrightarrow x_{[1]} \qquad e_1 \leftrightarrow x_{[2]}. \tag{11}$$

We also note that (7) has the simple solution

$$x_{n,m,p} = c + (e_1 - o_1)n + (e_2 - o_2)m + (e_3 - o_3)p.$$

4. Lax pair

An important property of the CAC definition of integrability is that it provides a method of constructing a Lax pair. The recipe was provided by Nijhoff in [8]: we consider the three maps on planes 12, 23, 31. The third direction is taken as auxiliary (spectral) and the system is linearized in the corresponding variable $x_{[3]}$ and its shifts. We also use notation $o_3 = \mu, e_3 = \lambda$.

Solving for $x_{[31]}$ from $Q(x, x_{[3]}, x_{[1]}, x_{[31]}; \lambda, \mu, e_1, o_1) = 0$ yields

$$x_{[31]} = \frac{\begin{array}{c} x_{[3]}[x x_{[1]}(o_1 - \mu) + x o_1(\lambda - \mu) + x_{[1]}(e_1 o_1 - \lambda\mu) + \lambda o_1(e_1 - \mu)] \\ + x x_{[1]}\mu(-e_1 + o_1) + x o_1 \mu(-e_1 + \lambda) + x_{[1]} e_1 \mu(-\lambda + o_1) \end{array}}{\begin{array}{c} x_{[3]}[x_{[1]}(-e_1 + \lambda) + x(-\lambda + o_1) + \lambda(-e_1 + o_1)] + x x_{[1]}(e_1 - \mu) \\ + x(e_1 o_1 - \lambda\mu) + x_{[1]} e_1(\lambda - \mu) + e_1 \lambda(o_1 - \mu) \end{array}}$$

and this is then linearized by introducing f, g by

$$x_{[3]} = \frac{f}{g} \qquad x_{[23]} = \frac{f_{[2]}}{g_{[2]}} \qquad x_{[31]} = \frac{f_{[1]}}{g_{[1]}} \tag{12}$$

resulting in

$$\begin{aligned} f_{[1]} &= \kappa_1 f[xx_{[1]}(\mu - o_1) + xo_1(\mu - \lambda) + x_{[1]}(\lambda\mu - e_1 o_1) + \lambda o_1(\mu - e_1)] \\ &\quad + \kappa_1 g[xx_{[1]}\mu(e_1 - o_1) + xo_1\mu(e_1 - \lambda) + x_{[1]}e_1\mu(\lambda - o_1)] \\ g_{[1]} &= \kappa_1 f[x_{[1]}(e_1 - \lambda) + x(\lambda - o_1) + \lambda(e_1 - o_1)] + \kappa_1 g[xx_{[1]}(\mu - e_1) \\ &\quad + x(\lambda\mu - e_1 o_1) + x_{[1]}e_1(\mu - \lambda) + e_1\lambda(\mu - o_1)]. \end{aligned}$$

Here the overall separation factor κ_i may contain $x, x_{[i]}$. We write this, and the corresponding equation obtained from $Q(x, x_{[2]}, x_{[3]}, x_{[23]}; e_2, o_2, \lambda, \mu) = 0$ in matrix form,

$$\phi = \begin{pmatrix} f \\ g \end{pmatrix} \qquad \phi_{[i]} = \begin{pmatrix} f_{[i]} \\ g_{[i]} \end{pmatrix} \qquad \phi_{[i]} = L_i \phi \tag{13}$$

with

$$L_i = \kappa_i (x_{[i]} + o_i)(x + \lambda) \begin{pmatrix} \mu & \mu e_i \\ -1 & -e_i \end{pmatrix} + \kappa_i (x_{[i]} + \lambda)(x + e_i) \begin{pmatrix} -o_i & -\mu o_i \\ 1 & \mu \end{pmatrix}. \tag{14}$$

Now from the consistency condition $(\phi_{[1]})_{[2]} = (\phi_{[2]})_{[1]}$ we get the matrix equation

$$\hat{L}_1 L_2 = \tilde{L}_2 L_1 \tag{15}$$

which is satisfied modulo the map (7), provided that the separation constants κ_i satisfy

$$\frac{\kappa_1 \tilde{\kappa}_2}{\hat{\kappa}_1 \kappa_2} = \frac{(x_{[2]} + e_1)(x + e_2)(x_{[2]} + \lambda)}{(x + e_1)(x_{[1]} + e_2)(x_{[1]} + \lambda)}. \tag{16}$$

A simple solution to this is

$$\kappa_i = \frac{1}{(x + e_i)(x + \lambda)} \tag{17}$$

leading to

$$L_i = \frac{x_{[i]} + o_i}{x + e_i} \begin{pmatrix} \mu & \mu e_i \\ -1 & -e_i \end{pmatrix} + \frac{x_{[i]} + \lambda}{x + \lambda} \begin{pmatrix} -o_i & -\mu o_i \\ 1 & \mu \end{pmatrix}. \tag{18}$$

A similarity transformation with $S = \left(\begin{smallmatrix} 1 & 0 \\ \mu^{-1} & 1 \end{smallmatrix}\right)$ simplifies this further to

$$L_i' = \begin{pmatrix} (\mu - e_i)\dfrac{x_{[i]} + o_i}{x + e_i} & \mu e_i \dfrac{x_{[i]} + o_i}{x + e_i} - \mu o_i \dfrac{x_{[i]} + \lambda}{x + \lambda} \\ 0 & (\mu - o_i)\dfrac{x_{[i]} + \lambda}{x + \lambda} \end{pmatrix}. \tag{19}$$

5. Summary

The ‘consistency around the cube’ definition of integrability is very transparent since it leads directly to the Lax pair. It also defines a clear search problem whose complete solution is, unfortunately, still beyond our computational abilities. The integrable model (7) or (10) presented in this paper is the first result in our search project. This model has several interesting features:

- It does *not* satisfy the tetrahedron property.
- It has *two* parameters in each direction.
- It is dual under interchange of variables and parameters.

Further properties of this model will be discussed elsewhere. One open question is whether (10) can be considered as a discretization of some known integrable continuous model. However, since the natural reduction of the present model is to (3), rather than any of the KdV-type models presented in section 1, it is doubtful that it is a discretization of any member in the KP-hierarchy.

In this general class of lattice models (multi-linear, consistent around a cube) there are probably still many other models to be found.

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